UDK 517.957

Ambarzumyan type theorem for a quadratic pencil of sturm–liouville operators with eigenparameter in the boundary conditions

Babajanov B. A., Babajanov A.B., Matkarimov M.R., Yakubov H.E.
Urgench State University

Annotation. In this paper, the classical Ambarzumyan's theorem for the regular Sturm-Liouville problem is extended to energy-dependent Sturm-Liouville problem with a spectral parameter depending on the boundary condition.

Keywords: Quadratic pencil of Sturm-Liouville operators; uniqueness theorem; Ambarzumyan's theorem; inverse problem; asymptotic of eigenvalues.

1. Introduction

In 1929, Ambarzumyan investigated the Sturm-Liouville operator with Neumann boundary conditions, and proved that if its spectrum consists of zero and infinitely many other square integers, then the potential is zero. From a historical viewpoint, the work of Ambarzumyan [1] was the first paper in the theory of inverse spectral problems associated with Sturm-Liouville operators. Ambarzumyan's theorem was generalized in many directions [2-6]. In [7], Ambarzumyan's theorem was extended to the quadratic pencil of the Sturm–Liouville operators with spectral parameter contained in the boundary conditions by adding an additional condition for the potential. Boundary value problems with spectral parameter in boundary conditions have received much attention in the recent research literature [8-10]. Various physical applications of such problems were found in [11].

In this paper we prove Ambarzumyan's theorem for the quadratic pencil of the Sturm–Liouville operators with spectral parameter contained in the boundary conditions without any additional condition for the potential.

Quadratic pencil of the Sturm–Liouville operators arises in various models of quantum and classical mechanics. For instance, to this form can be reduced the corresponding evolution equations (such as the Klein–Gordon equation [12, 13]) that are used to model interactions between colliding relativistic spinless particles. Another typical example is related to vibrations of mechanical systems in viscous media, see [14].

We consider the boundary-value problem

$$\begin{cases} -y'' + q(x)y + 2\lambda p(x)y = \lambda^2 y, & 0 \le x \le \pi, \\ y(0) + \lambda(h - hy(0)) = 0, & h \in R, \\ y'(\pi) = 0 \end{cases}$$

where λ is a spectral parameter and the functions $p(x) \in C^1[0,\pi]$ and $q(x) \in C[0,\pi]$ are real.

We denote by $\lambda_n,\ n \in Z,$ the spectrum of the problem (1). It is well known [15] that the sequence $\{\lambda_n : n = 0, \pm 1, \pm 2, \pm 3, ...\}$ satisfies the classical asymptotic form

$$\lambda_n = n + \frac{b_1(\pi)}{\pi n} + \frac{\gamma_n}{n},$$

where $b_1(\pi) = \int_0^\pi [q(x) + p^2(x)]dx$ and $\sum_n \gamma_n^2 < \infty$.

Let $\{n : n = 0, \pm 1, \pm 2, ...\}$ be spectrum of the problem

$$\begin{cases} -y'' = \lambda^2 y, & 0 \le x \le \pi, \\ y'(0) + \lambda(h - hy(0)) = 0, \\ y'(\pi) = 0. \end{cases}$$

The main results of this paper are as follows.

Theorem. If $\lambda_n = n,\ n \in Z$ then, $q(x) = p(x) = 0,\ \forall x \in [0,\pi]$.

REFERENCES

1. Ambarzumyan, V. A., (1929) Über eine frage der eigenwerttheorie, Zeitschrift für Physik, 53, 690- 695.

2. Borg G. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, Bestimmung der Differentialgleichung durch die Eigenwerte. Acta Math, 1946, 78: 1–96.

3. Carlson R. and Pivorachik V. N., (2007) Ambarzumian's theorem for trees, Electronic Journal of Differential Equations, 142, 1--9 .
4. Chern H. H., Law C. K. and Wang H. J., (2001) Extension of Ambarzumyans theorem to general boundary conditions, Journal of Mathematical Analysis and Applications, 263, 333-342.
5. Yang C.F. and Yang X.P., (2009) Some Ambarzumyan type theorems for Dirac operators, Inverse Problems, 25(9).
6. Yang C.F. and Yang X.P., (2011), Ambarzumyan's theorem for with eigenparameter in the boundary conditions, Acta Mathematica Scientia, 31(4) 1561-1568.
7. H. Koyunbakan and A. Bajalan. Inverse Sturm-Liouville Problem with Energy dependent potential. Communication in Mathematical Modeling and Applications CMMA 2, No. 1, 38-43 (2017).
8. Chuan-Fu Yang, N. P. Bondarenko, Xiao-Chuan Xu. An inverse problem for the Sturm-Liouville pencil with arbitrary entire functions in the boundary condition. Inverse Problems and Imaging, volume 14, No. 1, 2020, 153–169.
9. C.-F. Yang and X.-C. Xu, Ambarzumyan-type theorem with polynomially dependent eigenparameter, Math. Methods Appl. Sci., 38 (2015), 4411–4415.
10. G. Freiling and V. A. Yurko, Inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter, Inverse Problems, 26 (2010), 17pp.
11. Fulton C T. Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. Proc. Roy. Soc. Edinburgh, 1997, 77(A): 293–308
12. P. Jonas. On the spectral theory of operators associated with perturbed Klein–Gordon and wave type equations. J. Oper. Theory, 29(2):207–224, 1993.
13. B. Najman. Eigenvalues of the Klein–Gordon equation. Proc. Edinb. Math. Soc. (2), 26:181–190, 1983.
14. M. Yamamoto. Inverse eigenvalue problem for a vibration of a string with viscous drag. J. Math. Anal. Appl., 152:20–34, 1990.

15. Gasymov M. G. and Guseinov G. Sh., (1981) Determination of a diffusion operator from the spectral data, Doklady Akademic Nauk Azerbaijan SSR, 37(2), 19-23.