Development of students' thinking in the process of teaching mathematics by means of examples and counterexamples

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Abstract. The article examines the essence of the concepts "example", "counterexample", provides all kinds of examples and counterexamples that take place in mathematics, and argues how they can help develop the thinking of students.

Keywords: development of students' thinking, examples in mathematics, counterexamples in mathematics, teaching mathematics.

In modern conditions in education, priority is given to the developmental function, which requires the teacher to search for and use active teaching methods. Such an active method in teaching mathematics is the organization of educational and research activities of students. This activity can be organized in the course of solving special search and research problems.

The reader can learn more about the essence of search and research tasks, their classification and didactic functions in the learning process in our work [2].

In the process of working on a search and research task, the student considers various solutions, thereby demonstrating the work of convergent, divergent and critical thinking.

Studies [7, 8] highlight the following qualities of critical thinking: clarity, transparency, accuracy, correctness; relevance, involvement in the case; consistency, consistency, consistency; depth, completeness and originality; beauty and perfection; evidence, argumentation.

In mathematics, the following four logical formulas are most commonly used:

1. $\forall x(A(x) \Rightarrow B(x));$ 3. $\exists x(A(x) \wedge B(x));$

2. $\forall x(A(x) \Rightarrow \overline{B(x)});$ 4. $\exists x(A(x) \wedge \overline{B(x)}).$

Judgments 1) and 2) are refuted by counterexamples, and judgments 3) and 4) are proved by examples.

Examples and counterexamples are objects of the same nature and the strategy for their search does not depend on the content of the judgment, but is dictated by its structure.

N.A. Kurdyumova [3] studied the impact of examples and counterexamples on the achievement of the developmental goals of teaching mathematics and concluded that these didactic tools enhance the developmental function of the learning process in mathematics, as they allow the development of logical and critical thinking.

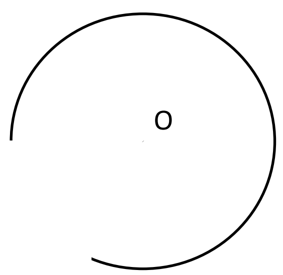
Student constructing counterexamples can be seen as a heuristic activity that goes through five phases of creative decision.

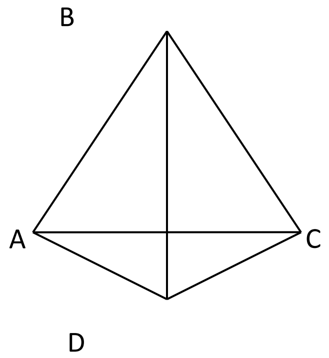
Students' activities to construct examples and counterexamples should be organized on various problems, including math. At first, it is advisable to invite students to build examples and counterexamples proving the falsity of statements for which the true meaning has already been indicated, and only then propose the construction of examples and counterexamples proving or refuting the truth or falsity of these statements or statements regarding which their true meanings are not indicated. Here are some examples.

I. The first series of examples is aimed at working with definitions of concepts, at bringing objects under the scope of concepts.

Give counterexamples proving the fallacy of the following definitions of concepts (table 1).

Table 1

	Definitions	**Counterexamples**
	2	**3**
	A circle is a plane curve, the points of which are equally distant from one point lying in this plane, called the center of the circle.	 All points of the specified flat line are removed from point O by the same distance, but the

		line is not a circle. The definition does not specify the essential feature "all".
	A rectangle is a rectangle whose diagonals are equal.	 Quadrilateral ABCD has equal diagonals AC and BD, but it is not a rectangle.

II. Give counterexamples proving the falsehood of the following statements (table 2). The table in the third column provides counterexamples.

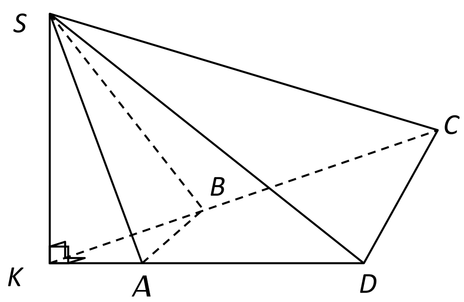
Table 2

	Statements	**Counterexamples**
	2	**3**
	Any number ending in one is divisible by 3.	41, the number ends in one, but it is not divisible by 3.

III. Give examples proving or refuting the following judgments (table 3). In the third column of the table, examples are given.

Table 3

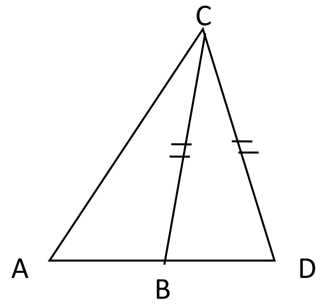
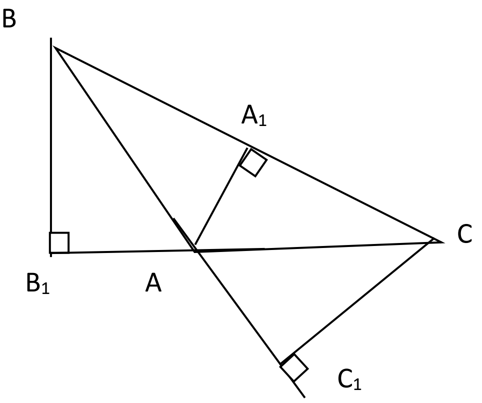
	Judgments	Examples

	2	**3**
	For any value of the variable x, the expression $$\sqrt{\frac{-(x-1)^2}{x^2+1}}$$ is meaningless.	An example refuting the judgment: for x=1, the expression makes sense and its value is zero.
	There is no quadrangular pyramid with two opposite faces perpendicular to the base.	 A counterexample refuting the statement. In the pyramid (see fig.) SABCD faces SBC and SAD are perpendicular to the base ABCD (The desired pyramid is obtained from the triangular pyramid SKCD in which the faces SCK and SDK are perpendicular to the base of KCD).

IV. Give counterexamples proving the falseness of these inferences (table 4).

Table 4

	Inferences	**Counterexamples**
	2	**3**
	If two sides and an angle opposite one of them, of one triangle are respectively equal to two sides and an angle opposite one of them, of the other triangle, then such triangles are equal	Counterexample confirming the falseness of the inference

		 In triangles ABC and ACB, side AC is common, sides CB and CD are equal, angle A is common, but triangles are not equal.
	The heights of the triangle intersect at one point	Counterexample to refute the inference  In obtuse triangle ABC, the heights BB_1, AA_1, CC_1 are drawn, do not intersect at all. If straight lines are drawn through the heights, then the latter will intersect at one point. The theorem should be formulated as follows: "Lines containing the heights of a triangle intersect at one point."

The tasks described in tables 1,2,3,4 are widely presented in our work [1] and in works [5, 6].

The next important stage in teaching students to work with examples and counterexamples should be the stage at which they are offered statements, statements, inferences about which it is not known whether they are false or true, and it is required to establish their truth values and give examples and counterexamples confirming these values. Here are some examples.

Example 1. When studying the concept of a prism, it is useful to invite students to answer the question: "Is this definition of a prism correct: a prism is a polyhedron whose two faces are equal polygons with correspondingly parallel sides, and all other faces are parallelograms?"

The answer to the question posed is negative and this is confirmed by Figure 1, which shows a polyhedron that is not a prism, but in which: ABCD and $A_1B_1C_1D_1$ are "bases", and all the other faces of the parallelogram. Note that such an erroneous definition of a prism was given earlier in the textbook by A.P. Kiselev.

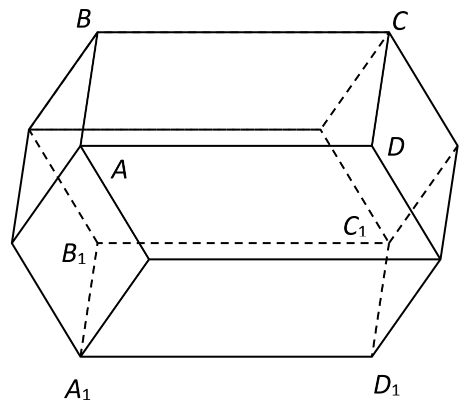


Fig.1

Example 2. Prove or refute the judgment: "For any $x > 0, \sqrt{x} < x$ "

Note that the refutation of a judgment requires the ability to build its negation.

As the experiment has shown, for the development of critical thinking of students, it is advisable to offer them tasks to find errors in problem solutions and theorem proofs. Problems of two types can be proposed:

a) tasks with conflicting information in the condition;

b) problems with conflicting information in the structure of the solution.

Working with problems containing conflicting information in the structure of the solution is structured as follows:

- the problem is proposed together with its solution;

- the error is included in the chain of logical conclusions;

- students are required to find the error in the solution and explain the reason for its occurrence.

The analysis showed that the greatest effect of counterexamples is achieved when two statements are formulated in which the condition and conclusion are interchanged, and the greater effect is achieved if the truth of the statements is unknown to the students.

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