On the problem of weak power conjugacy in a special class of Artin groups

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Abstract. The article gives an algorithm that allows solving the problem of weak power-law conjugacy of words in Artin groups, which are tree products of Artin groups with a tree structure and Artin groups of extra-large type.

Keywords: Artin group, tree structure, weak power conjugacy, diagram.

Let G' – be a finitely generated Artin group with co-representation

$$G' = \langle \sigma_1, \ldots, \sigma_n; <\sigma_i \sigma_j>^{m_{ij}} = <\sigma_j \sigma_i>^{m_{ji}}, i, j = \overline{1,n}, i \neq j \rangle,$$

where $<\sigma_i \sigma_j>^{m_{ij}}$ – consisting of m_{ij} alternating generators $\sigma_i, \sigma_j, i \neq j$, m_{ij} – element of the matrix (m_{ij}): $m_{ii} = 1, m_{ij} \geq 2$ or $m_{ij} = \infty, i \neq j$ [1].

The extra-large type of groups G' was introduced by K. Appel and P. Schupp [1]. It is typical for it that $m_{ij} > 3$ for all $i \neq j$

For G' we construct a graph Γ so that σ_i, $i = \overline{1,n}$, correspond to the vertices of the graph Γ, and $<\sigma_i \sigma_j>^{m_{ij}} = <\sigma_j \sigma_i>^{m_{ji}}, m_{ij} \neq \infty$,– an edge with ends σ_i, σ_j, $i \neq j$. G' has a tree structure if Γ – is a tree.

G', which has a tree structure, is a free product with amalgamation by cyclic subgroups of Artin groups on two generators.

The woody structure for G' in 2003 was identified by V.N. Bezverkhny [2].

Now

$$G = \langle \prod_{s=1}^{t} * G_s; \sigma_{i_m} = \sigma_{j_l}, i \neq j, i, j \in \{\overline{1,t}\} \rangle,$$

tree product G_s with tree structure or extra-large type, where the union G_i, G_j is taken over $\langle a_{i_m} \rangle$, $\langle a_{j_l} \rangle$, where a_{i_m} – is the generator of G_i, a_{j_l} – is the generator of G_j. G will be called an Artin group with a generalized tree structure or a special class of Artin groups.

It is known that in Artin groups of extra-large type, with a tree structure, and in a special class of Artin groups, the problems of equality and conjugacy of words are solvable [1] - [3].

Let an annular map *M*, be given on the plane, that is, its complement - two components. We assume that *K* is unbounded, and *H* is the bounded component of *M*, while $\partial M \cap \partial K$ is the outer, and $\partial M \cap \partial H$ is the inner boundary of *M*. The cycle σ of the smallest length, including the edges $\partial M \cap \partial K$, is the outer boundary cycle of *M*. Similarly, τ is the inner boundary cycle of *M*. Further, we denote a free product as *F*, R – symmetrized subset of *F*, $N = (R)^F$ – normal closure. If *u* and *z* are cyclically reduced in *F*, do not lie in *N*, are not conjugate in *F*, but conjugate in F/N, then there exists a reduced ring *R*-diagram *M* with at least one domain, and *σ*, τ are the inner and outer boundary cycles of *M*, $\varphi(\sigma) = u$, $\varphi(\tau) = z^{-1}$. Let *u*, *z* be two cyclically reduced words, $u, z \notin (R)^G$, are not conjugate in *F* and are conjugate in *G*. The boundary marks of the regions *D* of the *R*-diagram *M* are relations from *R*. Consider the transformations of the *R* – diagram of *M*. If D_1, D_2 intersect along the edge *e* so that the common label is either a relation from R_{ij}, or a single word in $F_{ij} = F_i * F_j$, then we erase *e*, $D_1$and D_2 are united into a single region, and in the latter case we cut out the resulting region by gluing it border. As a result, we obtain a reduced, that is, invariant under the considered transformations, ring *R*-diagram *M′* with boundary marks *u* and z^{-1}. In what follows, we will consider only such diagrams.

$\partial D \cap \partial M$ is a regular part of M if $\partial D \cap \partial M$ is the union of a sequence of closed edges occurring in the same order in ∂D and in ∂M. D is simple when $\partial M \cap \partial D$ is the correct part of ∂M. Further $d(D)$ is the number of edges of D; $d(v)$ is the degree of the vertex v; $i(D)$ is the number of internal edges, $||w||$ is the syllable length *w*. A simple domain D is Denov's [2] for $i(D) < 2$, the distance $\partial D \cap \partial M$ of the Denov's domain D is Denov's reduction M.

Definition 1 [2]. A sequence *П* of domains$D_1, D_2, \dots, D_n$ forms a strip in *M* with $\partial M = \gamma \cup \delta$, if:

1) $\forall i, 1 \le i \le n, \partial D_i \cap \gamma$ – connected path of syllable length at least 1;
2) $\forall i, 1 \le i < n$, boundaries D_i, D_{i+1} are crossed by *e*;
3) $||\partial D_1 \cap \gamma|| = ||\partial D_1 \cap \delta|| + 2, ||\partial D_n \cap \gamma|| = ||\partial D_n \cap \delta|| + 2$;
4) $||\partial D_j \cap \gamma|| = ||\partial D_j \cap \delta||, 2 \le j < n$.

Removing $\partial \Pi \cap \partial M$ in M is an $\bar{R}$-reduction [2].

Lemma 1 [2]. There is an algorithm that finds out for each cyclically reduced word *w* of the group *G* its *R*, $\bar{R}$ -irreducibility.

If *D* is located on both sides of the edge *e* and $\partial D = e\gamma e^{-1}\delta$, the glued edges *e* and e^{-1} intersect ∂D, then we have *(s – i)* – domain [2].

Theorem 1 [3]. If *M* is the *R*-conjugacy diagram of words $\phi(\sigma)$, $\phi(\tau) \in G$ without $(s - i)$-domains; moreover $\phi(\sigma)$, $\phi(\tau)$ are cyclically *R*, $\bar{R}$ -uncancellable, then *M* is one-layer.

The R-diagram M of conjugacy of cyclically R and $\bar{R}$-irreducible words $\varphi(\sigma), \varphi(\tau)$ from G is considered especially special if M has a unique D with $||\varphi(\partial D \backslash (\partial D \cap \sigma))| = |\varphi(\partial D \cap \sigma)||$. Replacing $\varphi(\sigma)$ by $\varphi(\tau)$ is a special ring R-cancellation [2].

Definition 2 [2]. w is dead-end if it is cyclically R, $\bar{R}$-irreducible and cannot be specially annular R-reduced.

It follows from [1] - [3] that one can effectively establish whether w is dead-end.

Lemma2. For conjugate dead-end words $v, w \in G$ their syllable lengths are equal and minimal in the sense that a word of shorter syllable length v is not conjugate to w.

The argument is obvious.

Definition 3 [2]. In G the weak power conjugacy problem is algorithmically solvable if, for any $v, w \in G, w \notin < v >$ the existence of an integer n, is effectively defined for which v, w^n are conjugate in G.

Lemma 3. Given any dead-end w from G, we can construct a word w_0 conjugate to it or its square, any power of which is R, $\bar{R}$-irreducible.

The proof is similar to the proof of lemma 11 in [4]. This is possible because we introduce the concept of a strip, which is identical to Artin groups with a tree structure, which is identical to that given in Definition 1, as well as the one-layer structure of the R-diagram M of conjugacy of words considered in theorem 1.

Theorem2. For any w from G one can construct a word w_0 conjugate to it or its square, any power of which is R, $\bar{R}$-irreducible.

Proof. Let us execute R, $\bar{R}$-reductions in cyclic w, then a special ring R-reduction, if any. We get the dead-end word conjugate to w. Next, we apply lemma 2 to the resulting word.

Lemma 3. For every cyclically irreducible v from G one can construct its cyclically conjugate R, $\bar{R}$-irreducible.

Theorem 3. In G, the weak power conjugacy problem is algorithmically solvable.

Proof. Suppose that w^n, v are conjugate in G. We must show that n is bounded.

Based on Theorem 2, we pass from w to w_0, each degree of which is R, $\bar{R}$-irreducible. Note that w_0 is conjugate to either w or its square.

Lemma 4. Let w_0^n and v_0 be conjugate. Then $n < (2|v_0| + 1)(|w_0| + |v_0| + 2)$.

Proof. It follows from [1], [5] that these problems have been solved in the factors of G.

The following cases are possible:

1. From w go to w_0.

Suppose that $w_0^n = z^{-1} v_0 z$, that is $w_0^n \sim {v_0}^2$.

Consider the ring diagram of M conjugacy of w_0^n *and* v_0 with $\partial M = \gamma \cup \delta, \varphi(\gamma) = w_0^n, \varphi(\delta) = v_0$.

According to theorem 1, M is a one-layer diagram. Let t denote the number of domains with access to γ and δ. The number of edges going to γ does not exceed $|v_0|$, the number of vertices does not exceed $|v_0| + 1$, then t, obviously, does not exceed the sum of edges and vertices, that is $2|v_0| + 1$. We have $n < |w_0^n| < t|r_0| \leq (2|v_0| + 1)|r_0|$, r_0 ∈R and has maximum length. Obviously, $|r_0| < |w_0| + |v_0| + 2$. Hence $n < (2|v_0| + 1)(|w_0| + |v_0| + 2)$.

2. From w^2 go to w_0. $w^n \sim v$. Then $(w^2)^n \sim v^2$. We replace w^2 by its conjugate w_0, any degree of which is cyclically R, $\bar{R}$-irreducible according to theorem 2. We obtain $w_0^n \sim v^2$. Replace v^2 by its equal in the group G R, $\bar{R}$-irreducible $v_0$according to lemma 3. Next, we use case 1.

The theorem is proved.

Note also that the special class of Artin groups considered in this paper belongs to the almost large Artin groups considered in [6]. Investigations of a special class of Artin groups by diagrammatic methods lead to simpler algorithms for solving the problems of equality, conjugacy, and weak power conjugacy. These methods are supposed to be used in the further study of these groups, in particular, to solve the problems of power conjugacy and intersection of cyclic subgroups.

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